Finite-size scaling and critical exponents in critical relaxation

Zhibing Li,[1,*] Lothar Schülke,[2] and Bo Zheng[2]
[1] International Centre for Theoretical Physics, I-34100 Trieste, Italy
[2] Universität-Gesamthochschule Siegen, D-57068 Siegen, Germany
(Received 31 August 1995)

We simulate the critical relaxation process of the two-dimensional Ising model with the initial state both completely disordered or completely ordered. Results of a method to measure both the dynamic and static critical exponents are reported, based on the finite-size scaling for the dynamics at the early time. From the time-dependent Binder cumulant, the dynamical exponent z is extracted independently, while the static exponents β/ν and ν are obtained from the time evolution of the magnetization and its higher moments.

PACS number(s): 02.70.Lq, 05.70.Jk, 64.60.Ht, 64.60.Fr

I. INTRODUCTION

For statistical systems in equilibrium or near equilibrium critical phenomena arise around the second-order phase transition points. Due to the infinite spatial and time correlation lengths there appear universality and scaling. The universal behavior of critical systems is characterized by the critical exponents. The determination of critical exponents has long been one of the main interests for both analytical calculations and numerical simulations.

Numerically critical exponents are usually measured by generating the configurations in the equilibrium with Monte Carlo methods. To obtain the critical exponents from the finite-size scaling, Binder's method is widely accepted [1, 2]. The dynamical exponent z is traditionally measured from the exponential decay of the time correlation for finite systems in the long-time regime [3, 4]. As is well known, numerical simulations near the critical point suffer from critical slowing down. Much effort has been made to circumvent this difficulty. To study the static properties of the system, some *nonlocal* algorithms, e.g., the cluster algorithm [5, 6], have proved to be very efficient compared to the normal *local* algorithms. However, in this case the original dynamic universality class is altered by the nonlocality of the algorithm. Properties of the original local dynamics cannot be obtained with nonlocal algorithms.

In recent years the exploration of critical phenomena has been broadened. Universality and scaling are also discovered for systems far from equilibrium. Better understanding has been achieved of the critical relaxation process even up to the early time. A representative example for such a process is that the Ising model initially in a random state with a small magnetization is suddenly quenched to the critical temperature and then evolves according to the dynamics of model A. Janssen, Schaub, and Schmittmann [7] have argued by an ϵ expansion up to two-loop order that, besides the well known universal behavior in the long-time regime, there exists another *universal* stage of the relaxation *at early times*, the so-called *critical initial slip*, which sets in right after the microscopic time scale t_{mic}. The characteristic time scale for the critical initial slip is $t_0 \sim m_0^{-z/x_0}$, where m_0 is the initial magnetization and x_0 is the dimension of it. It has been shown that x_0 is another independent critical exponent for describing the critical dynamic system.

The characteristic behavior of the critical initial slip is that, when a nonzero initial magnetization m_0 is generated, due to the anomalous dimension of the operator m_0, the time-dependent magnetization $M(t)$ undergoes a critical initial increase

$$M(t) \sim m_0 t^{\theta}, \tag{1}$$

where θ is related to x_0 by $x_0 = \theta z + \beta/\nu$. The exponent θ has been measured with Monte Carlo simulation for the Ising model and the Potts model both directly from the power-law increase of the magnetization in (1) [8, 9] and indirectly from the power-law decay of the autocorrelation [10]-[12]. The results are in good agreement with those from an ϵ expansion and the scaling relation is confirmed. For the two-dimensional Ising model, the numerical simulation gives $\theta = 0.191$ [10, 11, 13].

In a preceding paper [14] we proposed to measure both the dynamic and static exponents from the finite-size scaling of the dynamic relaxation at the early time. The idea is demonstrated for the two-dimensional Ising model. Since the measurement is carried out from the beginning of the time evolution, the method is efficient at least for the dynamic exponent z. Even though certain aspects of this dynamic approach should still be clarified, the results indicate a possible broad application of the short-time dynamics since the universal behavior of the dynamics at early time is found to be quite general [13, 15–19].

One of the purposes of the present paper is to give a detailed and complete analysis of the data briefly reported

*On leave of absence from Zhongshan University, 510275 Guangzhou, P.R. China.

in [14]. While in that Letter we have extracted the exponents by the optimal fit of two curves in a certain time interval ("global" fit), we propose here in addition another approach by which the critical exponents are obtained for each time step separately ("local" fit). Furthermore, the simulation has been extended to a longer time interval in order to confirm the stability of the measurements in the time direction and to see how the scaling possibly passes over to the long-time regime.

On the other hand, we may easily realize that for the scaling of the short-time dynamics a small initial magnetization is important besides the short initial correlation length. This is because $m_0 = 0$ is a fixed point for the renormalization-group transformation. However, there exists also another fixed point corresponding to $m_0 = 1$. Therefore one may like to know whether around that fixed point universality and scaling are also present or not. Actually some trials have been made with Monte Carlo simulation [20, 21]. For a large enough lattice, one may expect a power law decay of the magnetization

$$M(t) \sim t^{-\frac{\beta}{\nu z}} \tag{2}$$

before the exponential decay starts. From this behavior the exponent $\beta/(\nu z)$ can be estimated. However, the results are not yet conclusive.

Therefore another purpose of this paper is to present a systematic investigation of the finite-size scaling for the critical relaxation starting from $m_0 = 1$. It will be shown that scaling is observed in the early stage of the time evolution and with the help of the finite-size scaling all the critical exponents z, β, and ν can be obtained from the lattices which are much smaller that those in [20, 21].

Section II is devoted to the critical relaxation with $m_0 = 0$ and Sec. III to $m_0 = 1$. The final section contains some discussion.

II. THE CRITICAL RELAXATION WITH ZERO INITIAL MAGNETIZATION

The Hamiltonian for the Ising model is

$$-\mathcal{H}/k_B T = K \sum_{\langle ij \rangle} S_i \; S_j \;, \qquad S_i = \pm 1 \tag{3}$$

with $\langle ij \rangle$ representing nearest neighbors. In the equilibrium the Ising model is exactly solvable. The critical point locates at $K_c = \ln(1 + \sqrt{2})/2$, and the exponents $\beta = 1/8$ and $\nu = 1$ are known. In principle any type of the dynamics can be given to the system to study the nonequilibrium evolution processes. Unfortunately up to now none of them can be solved exactly.

In this paper we consider only the dynamics of model A. For the numerical simulation, typical examples are the Monte Carlo heat-bath algorithm and the Metropolis algorithm. For the analytical calculation, the Ising model should be assumed to be described by the $\lambda\phi^4$ theory. Then the Langevin equation can be introduced as a dynamic equation. For the Langevin dynamic system the renormalization-group method may be applied to understand the critical behavior as universality and scaling. For the critical relaxation with the initial condition of a very short correlation and small magnetization, Janssen, Schaub, and Schmittmann [7] have performed a perturbative renormalization calculation with an ϵ expansion up to two-loop order. They have obtained the scaling relation which is valid even in the short-time regime, and all the critical exponents including the new dynamic exponent θ which governs the initial behavior of the critical relaxation.

Of special interest here is the extension of the scaling form in Ref. [7] to finite-size systems [22, 8]. In accordance to the renormalization-group analysis for finite-size systems, after a microscopic time scale t_{mic} we expect a scaling relation to hold for the kth moment of the magnetization in the neighborhood of the critical point starting from the macroscopic short-time regime [23, 7, 24],

$$M^{(k)}(t, \tau, L, m_0) = b^{k\beta/\nu} M^{(k)}(b^z t, b^{-1/\nu}\tau, bL, b^{-x_0} m_0) \tag{4}$$

assuming that the initial correlation length is zero and the initial magnetization m_0 is small enough. Here t is the dynamic evolution time, $\tau = (T - T_c)/T_c$ is the reduced temperature, L is the lattice size, and b is the spatial rescaling factor. It has been discussed that under certain conditions the effect of m_0 remains even in the long-time regime of the critical relaxation [25]. This modifies the traditional scaling relation where the effect of m_0 has usually been suppressed.

In this paper we are only interested in the measurement of the well-known critical exponents z, β, and ν. To make the computation simpler and more efficient, we set m_0 to its fixed point $m_0 = 0$ in this section. Therefore the exponent x_0 will not enter the calculation. Furthermore, now the time scale $t_0 = m_0^{-z/x_0} \to \infty$, and the critical initial slip gets most prominent in time direction even though the magnetization itself will only fluctuate around zero.

The initial state with $m_0 = 0$ is prepared by starting from a lattice with all spins equal, then the spins of randomly chosen sites are switched, until exactly half of the spins are opposite. This initial state is updated with the heat-bath algorithm to 300 time steps for $L = 8$,16, and 32, and to 900 time steps for $L = 64$. The average over 50 000 samples of this kind with independent initial configurations has been taken in each run, and eight runs are used to estimate the errors. The critical value $K_c = 0.4406$ has been used and, in order to fix $1/\nu$ separately, we have repeated all simulations with $K = 0.4386$. In each case the observables $|M(t)|$, $M^{(2)}(t)$, and $M^{(4)}(t)$ have been measured.

To determine z *independently*, we introduce a *time-dependent* Binder cumulant

$$U(t, L) = 1 - \frac{M^{(4)}}{3(M^{(2)})^2}. \tag{5}$$

Here the argument τ has been set to zero and skipped. The simple finite-size scaling relation

$$U(t, L) = U(t', L'); \qquad t' = b^z t, \qquad L' = bL \tag{6}$$

for the cumulant is easily deduced from Eq. (4).

The exponent z can easily be obtained through searching for a time scaling factor b^z such that the cumulants from two different lattices in both sides in Eq. (6) collapse. We call this global scaling fitting. Actually, using the evolution time t as a scaling variable has been discussed in the determination of the dynamic exponent z with the Monte Carlo renormalization method [26]. The idea of measuring the exponents from the nonequilibrium state can also be found with respect to damage spreading [27, 13] and in the application to spin glass [28].

Here the cumulant $U(t, L)$ obtained from each of the eight runs for lattice size L has been compared with each run for $L' = 2L$. The best scaling factor 2^z has been estimated by the method of least squares. Figure 1 shows the cumulants for $L = 8$, 16, 32, and 64 by solid lines. The dots fitted to the lines for L up to 32 show the results for $L' = 2L$ rescaled in time with the best-fitting scaling factor 2^z. Only a selected number of 30 equidistant points has been plotted. One sees the remarkable scaling collapse even in the short-time regime. Compared with the figure shown in the preceding paper [14] the evolution time for $L' = 64$ has been extended up to $t'_{\rm max} = 900$. The average value for z and the error estimated from this procedure have been given in Table I for different pairs of lattices using Eq. (6). In the first steps of the time evolution, the values of $M^{(2)}$ and $M^{(4)}$ are quite small. A careful view of the data shows that their accuracy and in particular the accuracy of the cumulant $U(t, L)$ is not so good as for larger t. One may expect that skipping the region of smaller t will give more reliable results. Therefore we have performed fits for different time intervals $[t'_{\rm min}, t'_{\rm max}]$. The results in Ref. [14] correspond to $t'_{\rm min} = 1$ and $t'_{\rm max} = 300$. Figure 1 is from a fit with $t'_{\rm min} = 50$ and $t'_{\rm max} = 900$. The longer evolution time of $t'_{\rm max} = 900$ for lattice $L = 64$ shows also the stability of the scaling in the time direction. From the results we can see that z for larger time t is slightly bigger than that for smaller time t. Later we will come back to this point.

With z in hand, the scaling relation for the second moment

$$M^{(2)}(t, L) = b^{2\beta/\nu} M^{(2)}(t', L'); \qquad t' = b^z t, \qquad L' = bL, \tag{7}$$

can be used to estimate the exponent $2\beta/\nu$ in a similar way. The results have been included in Table I. The

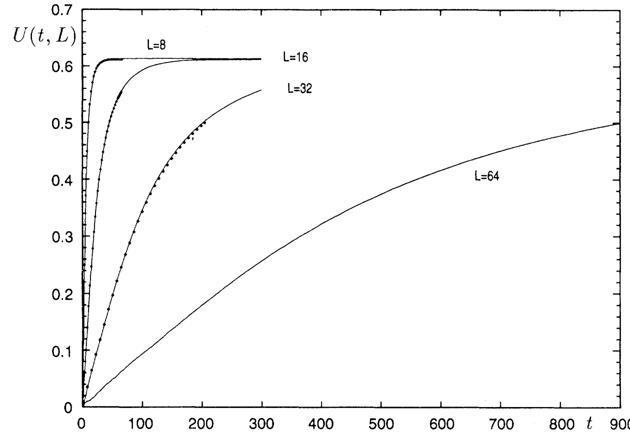

FIG. 1. The cumulants $U(t, L)$ for $L = 8$, 16, 32, and 64 for initial magnetization $m_0 = 0$ plotted versus the time t. The dots fitted to the lines show the cumulants with lattice size $2L$ rescaled in time by the best-fit value 2^z given in Table I.

curves for $M^{(2)}$ for the different lattice sizes and the corresponding scaling fits can be found in Fig. 2. We have cut the time scale at $t = 220$ in order to show the data relevant for the scaling fit more clearly.

Slightly more complicated appears the determination of $1/\nu$. One can use the derivative with respect to τ either of U or of $M^{(2)}$. Here the latter gives more stable results. From

$$\partial_\tau \ln M^{(2)}(t, \tau, L)|_{\tau=0} = b^{-1/\nu} \partial_{\tau'} \ln M^{(2)}(t', \tau', L')|_{\tau'=0} \tag{8}$$

with $t' = b^z t$ and $L' = bL$, the exponent $1/\nu$ can independently be calculated. The derivative is approximated by taking the difference of the values for $M^{(2)}$ at $K = K_c = 0.4406$ and $K = 0.4386$, divided by its value at K_c. It is clear and can also be seen in Fig. 3 that the result for this related quantity fluctuates more than that for $M^{(2)}(t, L)$ and $U(t, L)$, in particular for small t. The results of this calculation have also been included in the last column of Table I.

TABLE I. Results for z, $2\beta/\nu$, and $1/\nu$, respectively, from the two-dimensional Ising model with initial magnetization $m_0 = 0$. The values are obtained from a global scaling fit for two lattices.

Input	Lattice	$t'_{\rm min}$	$t'_{\rm max}$	z	$2\beta/\nu$	$1/\nu$
U	$8 \leftrightarrow 16$	10	300	2.100(2)	0.2473(02)	1.13(1)
	$16 \leftrightarrow 32$	10		2.149(2)	0.2494(06)	1.08(4)
M^2		50	300	2.134(4)	0.2510(11)	1.00(5)
	$32 \leftrightarrow 64$	200		2.140(4)	0.2488(11)	1.03(4)
		50	900	2.151(2)	0.2531(08)	1.02(2)
		200		2.153(2)	0.2523(08)	1.02(2)
$\tilde{U}$	$32 \leftrightarrow 64$	50	900	2.151(3)	0.2515(11)	1.03(2)
$\lvert M \rvert$		200		2.152(3)	0.2521(11)	1.03(2)

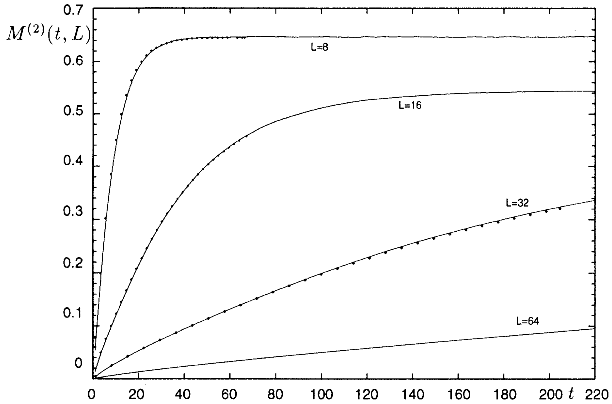

FIG. 2. The second moments $M^{(2)}(t, L)$ for $L = 8$, 16, 32, and 64 for initial magnetization $m_0 = 0$ plotted versus the time t. The dots fitted to the lines show the second moments with lattice size $2L$ rescaled in time by the best-fit value 2^z and $2\beta/\nu$ given in Table I.

It is interesting to point out, see Table I, that the results from the scaling fit of $L = 16$ and $L = 32$ are already quite good. This is probably due to the fact that the spatial correlation length in the short-time regime of the dynamic evolution is very small and therefore from small lattices one can already obtain reasonable results. We also like to mention that the procedure of comparing each run for the small lattice with each run for the big lattice may underestimate the errors, since these measurements are probably not completely independent. Therefore the data including the errors given in Table I do not always cover the exact values. The real errors may be a factor 2 or 3 bigger. For example, one may think of grouping the eight runs for the small lattice and the eight runs for the big lattice into eight independent pairs and carrying out the measurements with them.

In order to have more rigorous understanding of the dynamic scaling, we alternatively present a *local* approach for estimating the critical exponents, in contrast to the *global* scaling fitting procedure discussed above. For example, comparing the functions $U(t, L)$ and $U(t', 2L)$, for each time step t we search for t' such that $U(t, L) = U(t', 2L)$, and from the ratio $t'/t = 2^z$ the value of z is obtained according to Eq. (6). For the same time step t this particular value of z can be used to estimate $2\beta/\nu$ from Eq. (7) and $1/\nu$ from Eq. (8). Then we obtained all the exponents as functions of the time t. The result is shown in Fig. 4 for a pair of lattices with $L = 32$ and $L = 64$. In order to guide the eye, three horizontal lines $z = 2.14$, $2\beta/\nu = 0.25$, and $1/\nu = 1$ are included, the latter two values being the exact results for the two-dimensional Ising model. The figure shows clearly that the fluctuations for smaller time t are big. But for z and in particular for $2\beta/\nu$ the curve tends very nicely to a horizontal line for $t \gtrsim 30$. The situation is less satisfactory for the curve of $1/\nu$ where it shows still some fluctuations up to a fairly late time t. The reason may be either less statistics or from the approximation of the differentiation by a difference. The exponents z, $2\beta/\nu$, and $1/\nu$ can be obtained by averaging over the time direction. To show the effect of the fluctuations for smaller t, one takes the average starting from different initial times. The values of the exponents obtained in this way have been given in Table II. The errors have again been estimated by a comparison of each run for L and with each run for $L' = 2L$.

In Fig. 4, when $t \lesssim 30$ the exponent z is somewhat small. This might be due to the effect of $t_{\rm mic}$. In some cases, e.g., in the measurement of θ [8, 9, 29] and β/ν in the next section, this effect hardly shows up. However, in some other cases it can remain until $t \sim 20 - 30$ [9, 29]. This kind of effect probably comes from the fact that the

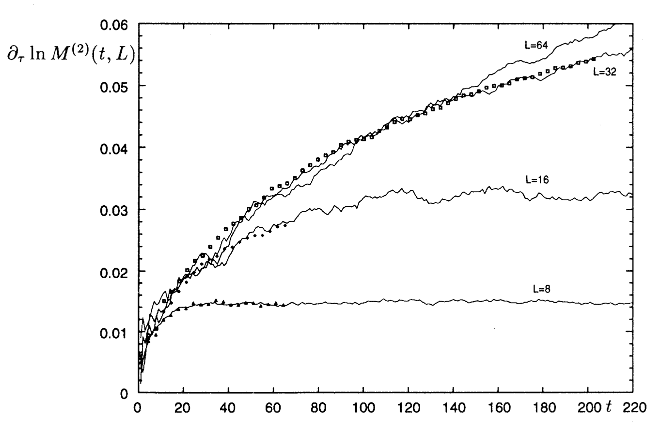

FIG. 3. $\partial_\tau \ln M^{(2)}(t, L)$ for $L = 8$, 16, 32, and 64 for initial magnetization $m_0 = 0$ plotted versus the time t. The dots fitted to the lines show those with lattice size $2L$ rescaled in time by the best-fit values 2^z and $1/\nu$ given in Table I.

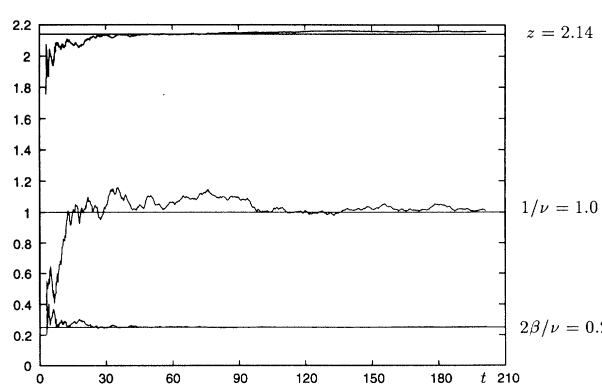

FIG. 4. The curves show the values for z, $2\beta/\nu$, and $1/\nu$ calculated for initial magnetization $m_0 = 0$ with the local scaling fit.

TABLE II. Results for z, $2\beta/\nu$, and $1/\nu$, respectively, from the two-dimensional Ising model with initial magnetization $m_0 = 0$. Values are from an average in time direction with a local scaling fit.

Input	Lattice	$t'_{\rm min}$	$t'_{\rm max}$	z	$2\beta/\nu$	$1/\nu$
U	8↔16	10	300	2.333(4)	0.2372(08)	1.11(1)
	16↔32	10	300	2.143(2)	0.2448(11)	1.19(5)
M^2	32↔64	50	900	2.148(3)	0.2510(16)	1.07(2)
		200		2.155(2)	0.2488(10)	1.05(2)
$\tilde{U}$	32↔64	50	900	2.144(4)	0.2541(28)	1.07(2)
$\|M\|$		200		2.153(3)	0.2501(14)	1.05(2)

initial magnetization density is not uniform enough. As we have already mentioned before, from Fig. 4 one can see explicitly even for $t \gtrsim 30$ that the exponent z slightly rises as time evolves. Even though the reason is not clear, it is interesting to note that the tendency of z to rise does not affect the measurement of the static exponents, especially $2\beta/\nu$. They are quite stable. On the other hand, even for z itself, 1% of the fluctuation in the time direction should also be not too bad. It may be due to the finite-size effect or some technical reasons.

The curves from a scaling fit of the lattices $8 \leftrightarrow 16$ and $16 \leftrightarrow 32$ which are not shown in the figures here look qualitatively the same as those in Fig. 4, but the fluctuations for $1/\nu$ are somewhat larger. The curve for z from the smaller lattice sizes $8 \leftrightarrow 16$, however, rises continuously for $t \gtrsim 20$, thus showing that such a lattice size is too small for this kind of analysis. The result of an average of these values is also included in Table II.

To measure z, $2\beta/\nu$, and $1/\nu$, instead of U, $M^{(2)}$, and $\partial_\tau M^{(2)}$, one can also use $\tilde{U}$, $|M|$, and $\partial_\tau |M|$ with

$$\tilde{U}(t,L) = 1 - \frac{M^{(2)}}{|M|^2}. \tag{9}$$

From Eq. (4) one easily deduces that the relation (6) holds for U as well as for $\tilde{U}$, and (8) holds as well for $\partial_\tau \ln |M|$. Only Eq. (7) is slightly modified to

$$|M(t,L)| = b^{\beta/\nu} |M(b^z t, L')|, \qquad L' = bL. \tag{10}$$

We do not plot the curves here, since they look very much the same as Figs. 1–4 for the global and local analysis. For simplification we have compared only the two lattice sizes $L = 32$ and $L = 64$, where we have used the full time scale up to $t'_{\rm max} = 900$ for $L = 64$. The results for the global fit have been included in the lower part of Table I. All the values in the table are remarkably consistent. The same holds for the results from the local approach. They are shown in Table II.

III. THE CRITICAL RELAXATION WITH INITIAL MAGNETIZATION ONE

In the preceding section we have investigated the finite-size scaling of the critical relaxation of the Ising model up to even the macroscopic short-time scale, starting from a random state with zero initial magnetization, i.e., a completely disordered state. From a measurement of the time evolution of the observables $|M(t,L)|$, $M^{(2)}(t,L)$, and $M^{(4)}(t,L)$ together with the scaling relation (4) the critical exponents z, $2\beta/\nu$, and $1/\nu$ were obtained. They are in good agreement with the known results. This is a strong support for the scaling relations derived by Janssen, Schaub, and Schmittmann [7].

At this stage one may ask whether there is also a scaling relation for the critical relaxation starting from a completely ordered state, i.e., with initial magnetization $m_0 = 1$. It has been known for some time that, before the exponential decay of the magnetization starts, there exists a time regime where the magnetization behaves nonlinearly and decays according to a power-law. The question is only *when* such a scaling behavior starts. Some effort has been made in this direction [20, 21] with Monte Carlo simulation. The authors have simulated the critical relaxation with an extremely big lattice but only up to a quite short evolution time and have estimated the exponent $\beta/(\nu z)$ from the power-law decay of $M(t)$. However, the result has not been so clear and also other

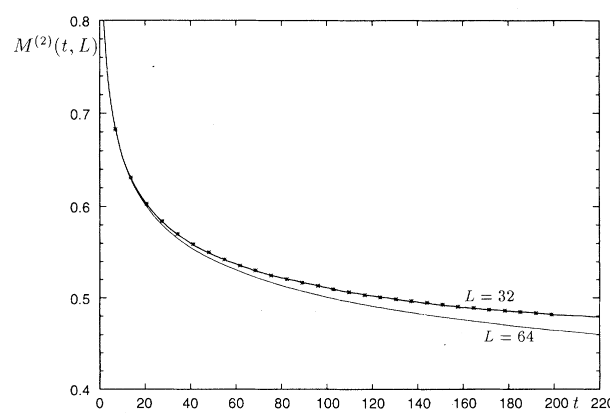

FIG. 5. The second moments $M^{(2)}$ for $L = 32$ and 64 for initial magnetization $m_0 = 1$ plotted versus time t. The dots fitted to the curve for $L = 32$ show the second moment with lattice size $L = 64$ rescaled in time by the best-fit value 2^z and $2\beta/\nu$ given in Table III.

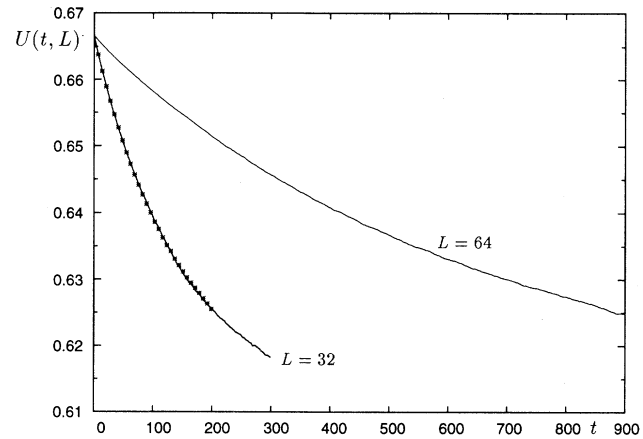

FIG. 6. The cumulants $U(t, L)$ for $L = 32$ and 64 for initial magnetization $m_0 = 1$ plotted versus time t. The dots fitted to the line of $L = 32$ show the cumulant with lattice size $2L$ rescaled in time by the best-fit value 2^z given in Table III.

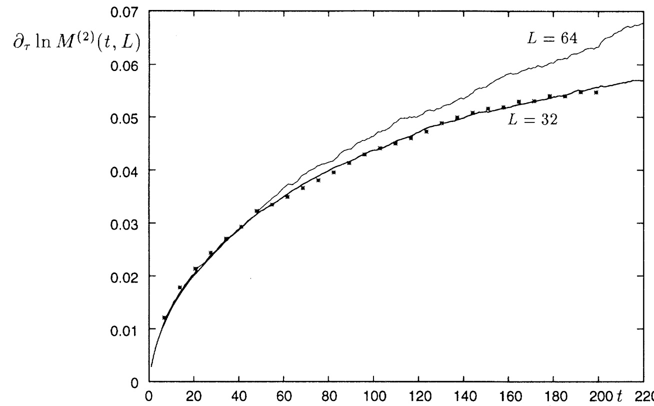

FIG. 7. $\partial_\tau \ln M^{(2)}(t, L)$ for $L = 32$ and 64 for initial magnetization $m_0 = 1$ plotted versus time t. The dots fitted to the line for $L = 32$ show that with lattice size $L = 64$ rescaled in time by the best-fit values 2^z and $b^{1/\nu}$ given in Table III.

exponents such as z and $1/\nu$ have not been obtained.

In this section we study systematically the scaling behavior of the critical relaxation from a completely ordered initial state, but in *finite systems*, following a procedure parallel to that discussed in the preceding section. The advantage is that a not too big finite system allows for longer evolution time even though the power-law decay of the magnetization will not be perfect. From our results we confirm that the scaling appears in a quite early stage of the relaxation as in the case of $m_0 = 0$.

Here we will only present data from the scaling collapse for lattice sizes $L = 32$ and $L = 64$. Also we confine ourselves to four runs with 50 000 updates each, instead of eight runs in the preceding section.

The curves for the second moment of the magnetization start with the value one for $t = 0$ and decrease for later time. This is seen in Fig. 5. A similar decrease is found for the curves for the cumulants $U(t, L)$ in Fig. 6, while $\partial_\tau \ln M^{(2)}(t, L)$ in Fig. 7 shows a rising behavior. In all three figures points mark the values for $L = 64$ rescaled in time by the best-fit values of z, $2\beta/\nu$, and $1/\nu$. Surprisingly here we also observe very nice dynamic scaling. Table III shows the results of the global fitting procedure up to $t'_{\rm max} = 300$ or $t'_{\rm max} = 900$, respectively. Since the values of U, $M^{(2)}$, and $\partial_\tau M^{(2)}$ for smaller t are large now compared with those in the case of $m_0 = 0$ in the preceding section, the results show fewer fluctuations for smaller time t. Actually one may also expect that due to the unique initial configuration less statistics is needed to obtain stable results. This is also supported by Fig. 8 from the local approach. It is very interesting that the exponent $2\beta/\nu$ shows almost *invisible* fluctuations in the whole time regime even up to the very beginning, although z has some similar unstable behavior as that in the case of $m_0 = 0$. Especially in the first 30 time steps its value is also a bit small. The fact that in the first steps of t the exponent z is quite near to 2.0 might indicate that at the very beginning of the time evolution the system is "classical." Similarly to Table II of the preceding section, Table IV gives the averages over the time direction, starting at different initial values $t'_{\rm min}$.

As in the preceding section, we carry out also the analysis with $\tilde{U}$ defined in (9), $|M|$ and $\partial_\tau |M|$. The results have been included in the lower parts of Table III from the global fit, or in Table IV from the local approach, respectively. They again have a similar quality as those reported above. In comparison with the results in [20, 21, 30], the value of z we obtained is definitely smaller and

TABLE III. Results for z, $2\beta/\nu$, and $1/\nu$, respectively, from the two-dimensional Ising model with initial magnetization $m_0 = 1$. The values are obtained from a global scaling fit for two lattices.

Input	Lattice	$t'_{\rm min}$	$t'_{\rm max}$	z	$2\beta/\nu$	$1/\nu$
U		50	300	2.121(4)	0.2489(4)	1.03(2)
	32↔64	200		2.122(5)	0.2491(5)	1.02(2)
M^2		50	900	2.129(5)	0.2503(5)	1.04(2)
		200		2.129(5)	0.2505(6)	1.04(2)
$\tilde{U}$	32↔64	50	900	2.140(5)	0.2514(6)	1.07(2)
$\lvert M\rvert$		200		2.141(5)	0.2515(7)	1.07(2)

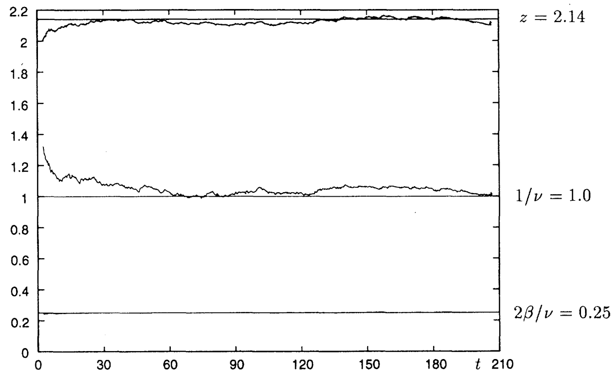

FIG. 8. The curves show the values of z, $2\beta/\nu$, and $1/\nu$ obtained for initial magnetization $m_0 = 1$ from a local scaling fit for $L = 32$ and $L = 64$.

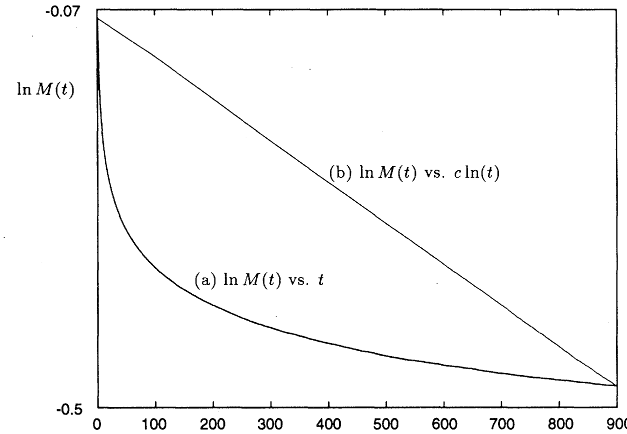

FIG. 9. The quantities $\ln M(t, L)$ for $L = 64$ and $m_0 = 1$ plotted (a) versus t and (b) versus $c\ln(t)$. The factor c has been chosen such that at 900 of the abscissa both curves coincide.

near to the one from the ϵ expansion and other traditional measurements [4, 31]. In case of $m_0 = 0$, the results for z measured from U and $\tilde{U}$ are almost the same. However, in case of $m_0 = 1$, the value measured from U is somewhat smaller than that measured from $\tilde{U}$. In the construction of U we have not subtracted the odd moments. This may have some effect on the measurement of z.

Here we would like to point out that the exponent $1/\nu$ measured in this section as shown in Table III and Table IV (as well as in Table I and Table II) is somewhat bigger than the exact value $1/\nu = 1.0$. This is probably due to the approximation of the differentiation by the difference. To improve this situation, either a simulation with a smaller difference in the coupling K should be carried out, or one expresses the derivative in terms of higher moments [32].

As compared to those in the preceding section for the case of $m_0 = 0$, the results here are somewhat more stable. This may really indicate that it is also promising to measure the critical exponents from the critical relaxation process starting from an ordered state even though some more theoretical arguments like that by Janssen, Schaub, and Schmittmann [7] are still needed. It is clear that in case of a random initial state with $m_0 = 0$ all the observables discussed start their evolution from zero and therefore the fluctuations at the beginning of the time evolution are naturally bigger. Besides this, the effect of the nonuniformity of the magnetization density in the practically generated initial configurations is not completely negligible for not too big lattices. Another way to measure the critical exponents is to study the critical relaxation starting from an initial state with small but nonzero initial magnetization [9, 29]. However, the situation is not so clear since the new dynamic exponent θ enters the calculation. Further investigation is needed.

Finally we plot the time evolution of the magnetization with both double-log scale and semilog scale in order to see whether it has entered the regime of linear decay or not. In Fig. 9 the straight line shows definitely that the magnetization is still in the regime of nonlinear decay. From the slope of $M(t)$ in double-log scale one can obtain the exponent $\beta/(\nu z)$. For each time t we have measured it by a least-squares fit within a time interval $[t, t + 50]$. In Fig. 10 instead of $\beta/(\nu z)$ the exponent z is plotted vs time using the exact value $\beta/\nu = 1/8$. Note that the time scale in Fig. 10 is different from that in Fig. 8. Figure 10 shows that in the regime $150 \gtrsim t \gtrsim t_{\rm mic} \approx 30$ the values for z are rather consistent with those obtained before, especially those measured from $\tilde{U}$ in Table III. However, the lattice size $L = 64$ seems not to be big enough to

TABLE IV. Results for z, $2\beta/\nu$, and $1/\nu$, respectively, from the two-dimensional Ising model with initial magnetization $m_0 = 1$. Values are obtained from the average in the time direction from $t'_{\rm min}$ up to $t'_{\rm max} = 900$ with a local scaling fit.

Input	Lattice	$t'_{\rm min}$	z	$2\beta/\nu$	$1/\nu$
U	32↔64	50	2.122(7)	0.2508(3)	1.04(2)
M^2		200	2.122(8)	0.2510(4)	1.04(2)
$\tilde{U}$	32↔64	50	2.133(6)	0.2516(4)	1.06(2)
$\|M\|$		200	2.134(7)	0.2520(5)	1.06(3)

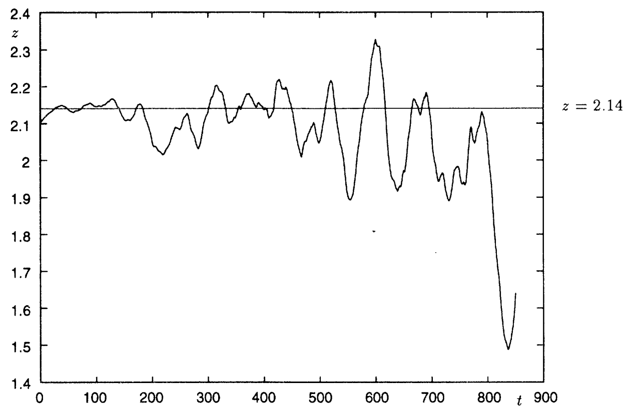

FIG. 10. The exponent z calculated from the slope of the magnetization $M(t)$ in a double-log scale. The slope $\beta/(\nu z)$ has been fitted for each time step within the time interval $[t, t+50]$. In order to calculate z we have used $\beta/\nu = 1/8$ as input.

present a rigorous power-law behavior in the whole time region.

IV. DISCUSSION

We numerically simulate the critical relaxation process of the two-dimensional Ising model with the initial state both completely disordered or completely ordered. Based on the finite-size scaling for the dynamics at the early time, both the static and dynamic critical exponents are measured. To determine z independently, a time-dependent Binder cumulant is constructed. The value of z measured from the critical relaxation from a completely ordered state is slightly smaller than that from a completely disordered state. The reason is not yet clear. Taking the average of the four measurements of z from the global scaling fit of U and $\tilde{U}$ within the time interval $[50, 900]$ of t' for both relaxation processes, see Table I and Table III, we get

$$z = 2.143(5).$$

This should be compared with the existing numerical results $z = 2.13(8)$ from [4], $z = 2.14(5)$ from [31], and $z = 2.13(1)$ from [26], and also with $z = 2.126$ obtained from an ϵ expansion in [33], even though the value of z is still a matter of controversy [34, 35], and especially some bigger values are also reported recently [20, 21, 30, 36, 13]. It is remarkable that from the short-time dynamics one cannot only efficiently measure the dynamic exponent z, but also the static exponents. Especially the quality of the exponent $2\beta/\nu$ is very good. All these results provide strong confirmation for the scaling relation at the early time of the critical relaxation process. Compared with the traditional methods, the advantage of our *dynamic* Monte Carlo algorithm is that the measurement is carried out in the beginning of the time evolution rather than in the equilibrium where critical slowing down is more severe. Therefore our method is efficient. Compared with the nonlocal algorithms, our dynamic algorithm can study the properties of the original local dynamics. On the other hand, it has recently been suggested that the critical exponents can also be measured from the *power-law* behavior of the observables including the autocorrelation in the macroscopic short-time regime in a large enough lattice [9, 29]. Compared with that approach, the advantage of estimating the exponents from the dynamic finite-size scaling as reported in this paper is that one needs not too big lattices. However, the result has to be obtained by comparing two lattices and longer time of the evolution for the bigger lattice should be carried out.

It is somewhat surprising that for the critical relaxation from the completely ordered initial state there exist also universality and scaling in such an early stage of the time evolution. Further investigation especially on a more general critical relaxation process from an ordered state with initial magnetization m_0 smaller but near to one can be interesting. One might expect that a new dynamic exponent should be introduced in order to complete the scaling relation.

ACKNOWLEDGMENTS

One of the authors (Z.B.L.) is grateful to the Alexander von Humboldt-Stiftung for financial support. The authors would like to thank K. Untch for the help in maintaining our workstations.

[1] K. Binder and D. W. Heermann, Monte Carlo Simulation in Statistical Physics. (Springer, Berlin, 1992).
[2] K. Binder, Z. Phys. B **43**, 119 (1981).
[3] S. Wansleben and D. P. Landau, Phys. Rev. B **43**, 6006 (1991).
[4] J. K. William, J. Phys. A **18**, 49 (1985).
[5] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **58**, 86 (1987).
[6] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).
[7] H. K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B **73**, 539 (1989).
[8] Z. B. Li, U. Ritschel, and B. Zheng, J. Phys. A **27**, L837 (1994).
[9] L. Schülke and B. Zheng, Phys. Lett. A **204**, 295 (1995).
[10] D. A. Huse, Phys. Rev. B **40**, 304 (1989).
[11] K. Humayun and A. J. Bray, J. Phys. A **24**, 1915 (1991).
[12] N. Menyhárd, J. Phys. A **27**, 663 (1994).
[13] P. Grassberger, Physica **A 214**, 547 (1995).
[14] Z. B. Li, L. Schülke, and B. Zheng, Phys. Rev. Lett. **74**, 3396 (1995).
[15] K. Oerding and H. K. Janssen, J. Phys. A **26**, 3369 (1993); **26**, 5295 (1993).
[16] K. Oerding and H. K. Janssen, J. Phys. A **27**, 715 (1994).
[17] A. J. Bray, Adv. Phys. **43**, 357 (1994).

[18] U. Ritschel and P. Czerner, Phys. Rev. Lett. **75**, 3882 (1995).
[19] A. J. Bray and B. Derrida, Phys. Rev. E **51**, R1633 (1995).
[20] D. Stauffer, Physica A **186**, 197 (1992).
[21] C. Münkel, D. W. Heermann, J. Adler, M. Gofman, and D. Stauffer, Physica A **193**, 540 (1993).
[22] H. W. Diehl and U. Ritschel, J. Stat. Phys. **73**, 1 (1993).
[23] V. Privmann and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).
[24] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1989).
[25] U. Ritschel and H. W. Diehl, Phys. Rev. E **51**, 5392 (1995).
[26] M.-D. Lacasse, J. Vinals, and M. Grant, Phys. Rev. B **47**, 5646 (1993).
[27] G. G. Batrouni and A. Hansen, J. Phys. A **25**, L1059 (1992).
[28] R. E. Blundell, K. Humayun, and A. J. Bray, J. Phys. A **25**, L733 (1992).
[29] K. Okano, L. Schülke, K. Yamagishi, and B. Zheng (unpublished).
[30] B. Damman and J. D. Reger, Europhys. Lett. **21**, 157 (1993).
[31] S. Tang and D. P. Landau, Phys. Rev. B **36**, 567 (1987).
[32] A. M. Ferrenberg and D. P. Landau, Phys. Rev. B **44**, 5081 (1991).
[33] R. Bausch, V. Dohm, H. K. Janssen, and R. K. P. Zia, Phys. Rev. Lett. **47**, 1837 (1981).
[34] D. P. Landau, S. Tang, and S. Wansleben, J. Phys. Colloq. **49**, C8–1525 (1988).
[35] N. Ito, Physica A **196**, 591 (1993).
[36] M. Siegert and D. Stauffer, Physica A **208**, 1 (1994).